Chain Rule

In the one-variable case, we use the Chain Rule when we want to differentiate a composite function:

if you have $g(h(t))$,

$$\boxed{\frac{d}{dt}\left(g(h(t))\right) = g'(h(t)) \cdot h'(t)}, \quad \text{or equivalently}$$

$$\boxed{\frac{dg}{dt} = \frac{dg}{dh}\frac{dh}{dt}}$$

Example Compute $\frac{d}{dt}(e^{t^2})$.

Note $e^{t^2} = g(h(t))$, where $h(t) = t^2$ and $g(h) = e^h$.

So $\frac{dg}{dt} = \frac{dg}{dh}\frac{dh}{dt} = e^h \, 2t = e^{t^2}(2t) = 2te^{t^2}$.

Example Compute $\frac{d}{dt}(\cos(\sin t))$.

Note $\cos(\sin t) = g(h(t))$, where $h(t) = \sin t$, $g(h) = \cos h$.

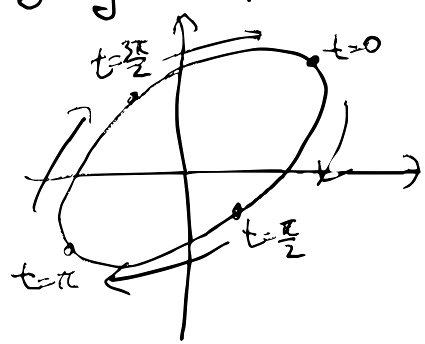
So $\frac{dg}{dt} = \frac{dg}{dh}\frac{dh}{dt} = -\sin h \cdot \cos t = -\sin(\sin t) \cdot \cos t$.

There is also a Chain Rule for several variable functions.

This is needed if you have a composition of functions involving several variables.

<u>Example</u> Consider the parametric equation of an ellipse,

$\vec{r}(t) = \langle x, y \rangle = \langle 2\cos t + \sin t, 2\cos t - \sin t \rangle$, representing a particle moving along the ellipse.



Then, consider the distance between the origin $(0,0)$ and the particle $\vec{r}(t)$ at time t. This is simply

$$|\vec{r}(t)| = \sqrt{(2\cos t + \sin t)^2 + (2\cos t - \sin t)^2}.$$

This function is the composition

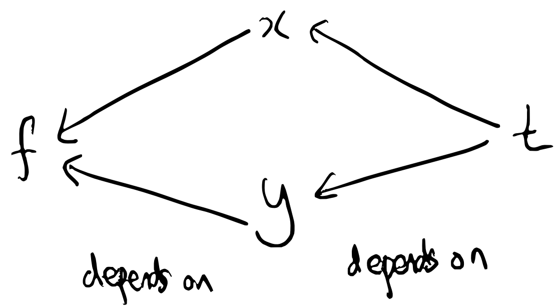
$d(x(t), y(t))$, where

$d(x,y) = \sqrt{x^2+y^2}$,

$x = 2\cos t + \sin t$
$y = 2\cos t - \sin t$.

Here, you plug two different (usual) functions $x(t), y(t)$ in the same variable into a function $f(x,y)$ of 2 variables. Let's call this

Case 1. It is helpful to think in layers:



Each arrow shows dependency: f depends on x, y, x depends on t, y depends on t.

In this case, $$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}}.$$

How to think I: From the previous lecture, we had

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy.$$ Divide by dt, we get

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt},$$ which is what we want!

How to think II: From the dependency diagram

$$\begin{tikzcd}
 & x \arrow[dl] & \\
f & & t \arrow[ul] \arrow[dl] \\
 & y \arrow[ul] &
\end{tikzcd}$$

there are two paths to go from t to f.

$$\begin{tikzcd}
 & x \arrow[dl] & \\
f & & t \arrow[ul]
\end{tikzcd} \quad , \quad \begin{tikzcd}
f & & t \arrow[dl] \\
 & y \arrow[ul] &
\end{tikzcd}$$

For each arrow, you associate the partial derivatives, and add the two possible paths:

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\begin{tikzcd}
 & x \arrow[dl] & \\
f & & t \arrow[ul]
\end{tikzcd} \qquad \begin{tikzcd}
f & & t \arrow[dl] \\
 & y \arrow[ul] &
\end{tikzcd}$$

<u>Example</u> If $f(x,y) = x^2 + 2xy$,

$x(t) = 2t^3$,

$y(t) = t^2$,

then $f(x(t), y(t)) = x(t)^2 + 2x(t)y(t)$

$= (2t^3)^2 + 2 \cdot (2t^3) \cdot t^2$

$= 4t^6 + 4t^5$.

So $\frac{df}{dt} = 24t^5 + 20t^4$. On the other hand,

$\frac{\partial f}{\partial x} = 2x + 2y$, $\frac{\partial f}{\partial y} = 2x$, $\frac{dx}{dt} = 6t^2$, $\frac{dy}{dt} = 2t$, so

$$\begin{aligned}\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= (2x+2y)\,6t^2 + 2x\cdot(2t) \\ &= 6t^2(4t^3 + 2t^2) + 4t^3\cdot(2t) \\ &= 24t^5 + 12t^4 + 8t^4 \\ &= 24t^5 + 20t^4.\end{aligned}$$

<u>Example</u> We can find the distance between a circle and a point using this. Let's say we have a circle

$$x(t) = \cos t, \quad y(t) = \sin t$$

and a point $(3,0)$.

Then, the distance between the point $(3,0)$ and the point (x,y) is $f(x,y) = \sqrt{(x-3)^2 + y^2}$, and the distance between the point $(3,0)$ and the point $(\cos t, \sin t)$ on the circle is $f(x(t), y(t))$.

So $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

$$= \frac{x-3}{\sqrt{(x-3)^2+y^2}}(-\sin t) + \frac{y}{\sqrt{(x-3)^2+y^2}} \cdot \cos t$$

$$= \frac{-(\cos t-3)\sin t + \sin t \cos t}{\sqrt{(\cos t-3)^2+\sin^2 t}}$$

$$= \frac{3\sin t}{\sqrt{(\cos t-3)^2+\sin^2 t}}.$$

Recall that, in Calculus, to find minimum/maximum, you have to find critical points, namely $\frac{df}{dt}=0$.

Calculus Reminder

Given a function $f(t)$, a critical point is a value of t such that $f'(t)=0$. The global minimum is the least value that $f(t)$ can take (where t can be any number). This may not exist, if $f(t)$ can be as negative as you want.

Example $f(t)=t$ has no global minimum

$f(t)=t^2$ has global minimum 0, because $t^2 \geq 0$.

If the global minimum exists, it is a critical point.

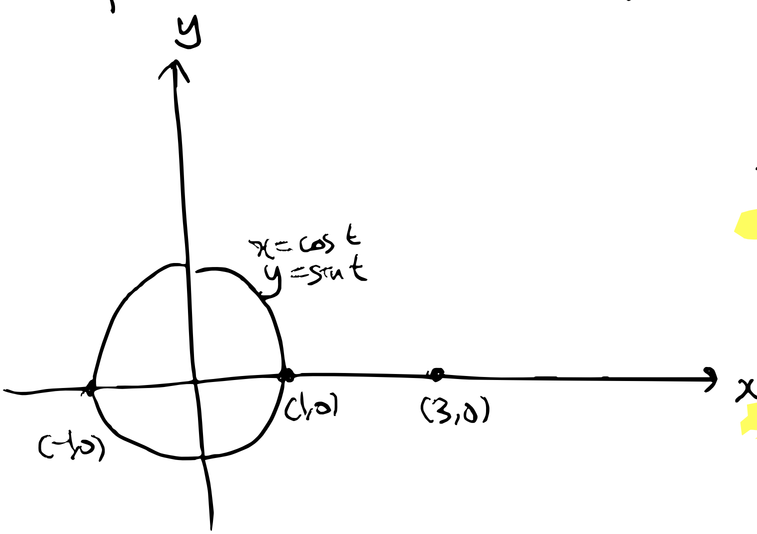
(Caveat: We're dealing with the case where $f(t)$ is everywhere defined. The situation gets more complicated when you have a different domain. This is a subject of later lectures.)

To find the global minimum, you thus need to find the critical points and check whether each of them achieves the global minimum or not.

<u>Example</u> $f(t) = t^2$ has critical point when $f'(t) = 0$, or $2t = 0$, so $t = 0$, as desired.

The same discussion applies to global maximum.

The calculation suggests that $\frac{df}{dt} = 0$ if $\sin t = 0$, or if t is an integer multiple of π, or $\cos t$ is either $+1$ or -1.



$\cos t = 1$ corresponds to $(1,0)$. the closest point.

$\cos t = -1$ corresponds to $(-1,0)$, the farthest point.

Case 2.

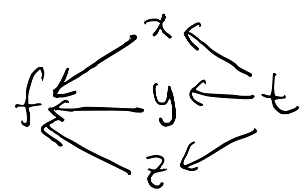
Similarly, if you have a 3-variable function $f(x,y,z)$ and three functions on t, $x(t), y(t), z(t)$, then there is a Chain Rule for $\frac{d}{dt}$ of $f(x(t),y(t),z(t))$.

$$\boxed{\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.}$$

How to think I: From $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$, after you divide by dt, you would get

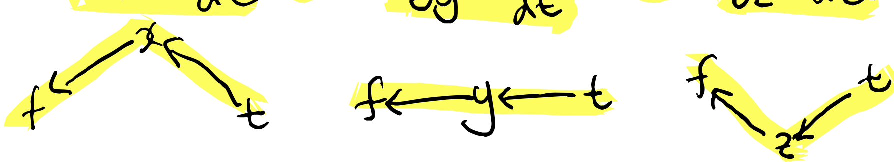
$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}.$$

How to think II: From



$\frac{df}{dt}$ is the sum of all paths from t to f.

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt}.$$



Example Let $f(x,y,z) = \dfrac{x+z}{y}$, and

$$x(t) = t^2 \qquad \frac{dx}{dt} = 2t \qquad \frac{\partial f}{\partial x} = \frac{1}{y}$$

$$y(t) = t+1. \quad \text{Then,} \quad \frac{dy}{dt} = 1 \qquad \frac{\partial f}{\partial y} = -\frac{x+z}{y^2}.$$

$$z(t) = t \qquad \frac{dz}{dt} = 1 \qquad \frac{\partial f}{\partial z} = \frac{1}{y}$$

Note that $f(x(t), y(t), z(t)) = \dfrac{x(t)+z(t)}{y(t)} = \dfrac{t^2+t}{t+1} = t$, so

$\dfrac{df}{dt} = 1$. On the other hand,

$$\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = \frac{1}{y}\cdot 2t + \left(-\frac{x+z}{y^2}\right)\cdot 1 + \frac{1}{y}\cdot 1$$

$$= \frac{2t}{t+1} - \frac{t^2+t}{(t+1)^2} + \frac{1}{t+1} = \frac{2t+1}{t+1} - \frac{t}{t+1} = \frac{t+1}{t+1} = 1.$$

Example We can also recover the distance between a point and a line as follows.

Suppose we have a line

$$L: \quad x = -1+2t, \quad y = -2-2t, \quad z = 3+t$$

and a point $P = (2,-1,-1)$.

Then, the distance between $P = (2,-1,-1)$ and a point $(-1+2t, -2-2t, 3+t)$ is $f(x(t), y(t), z(t))$,

where $f(x,y,z) = \sqrt{(x-2)^2+(y+1)^2+(z+1)^2}$,

$$x(t) = -1+2t,$$
$$y(t) = -2-2t,$$
$$z(t) = 3+t.$$

So, $\frac{dx}{dt} = 2$, $\frac{dy}{dt} = -2$, $\frac{dz}{dt} = 1$,

$$\frac{\partial f}{\partial x} = \frac{x-2}{\sqrt{(x-2)^2+(y+1)^2+(z+1)^2}}, \quad \frac{\partial f}{\partial y} = \frac{y+1}{\sqrt{(x-2)^2+(y+1)^2+(z+1)^2}},$$

$$\frac{\partial f}{\partial z} = \frac{z+1}{\sqrt{(x-2)^2+(y+1)^2+(z+1)^2}}.$$ So,

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$$= \frac{2(x(t)-2) - 2(y(t)+1) + (z(t)+1)}{\sqrt{(x(t)-2)^2+(y(t)+1)^2+(z(t)+1)^2}}$$

$$= \frac{2(2t-3) - 2(-1-2t) + (4+t)}{\sqrt{(2t-3)^2+(-1-2t)^2+(4+t)^2}} = \frac{9t}{\sqrt{(2t-3)^2+(-1-2t)^2+(4+t)^2}}$$

So, $\frac{df}{dt}=0$ if $t=0$, or the global minimum of distance $f(x(t),y(t),z(t))$ is $f(x(0),y(0),z(0))$

$$= \sqrt{3^2+1^2+4^2} = \sqrt{26}.$$

On the other hand, we know the distance between L & P is $|\vec{PQ} - \text{proj}_{\vec{v}} \vec{PQ}|$, where L passes Q and has directional vector $\vec{v}$. We can take $Q = (-1,-2,3)$ and $\vec{v} = \langle 2,-2,1\rangle$. Then $\vec{PQ} = \langle -3,-1,4\rangle$, and

$$\text{proj}_{\vec{v}} \vec{PQ} = \frac{\vec{PQ}\cdot\vec{v}}{|\vec{v}|^2}\vec{v} = \frac{\langle -3,-1,4\rangle\cdot\langle 2,-2,1\rangle}{|\langle 2,-2,1\rangle|^2}\langle 2,-2,1\rangle = 0,$$

so the distance is $|\vec{PQ}| = |\langle -3,-1,4\rangle| = \sqrt{3^2+1^2+4^2} = \sqrt{26}.$

Case 3 One can also compose two-variable functions with two-variable functions.

Namely, if you start with a two-variable function $f(x,y)$, and if x,y can be expressed as two-variable functions $x(s,t)$, $y(s,t)$, you get the composite function $f(x(s,t),y(s,t))$, which is a two-variable function in variables s,t.

Dependency diagram:

$$\begin{tikzcd}
& x \arrow[dl] & s \arrow[l] \arrow[ddl] \\
f & & \\
& y \arrow[ul, "\text{depends on}"'] & t \arrow[l, "\text{depends on}"'] \arrow[uul]
\end{tikzcd}$$

Then, the partial derivatives of the composite function

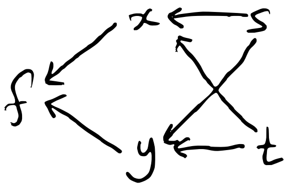
$f(x(s,t), y(s,t))$ are

$$\boxed{\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}}, \quad \boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

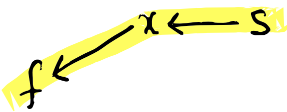
How to think I: From $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$, you can divide by ds and by dt to get

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds}, \quad \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

How to think II: $\frac{\partial f}{\partial s}$ is the sum over all possible paths from s to f:



has two paths from s to f,



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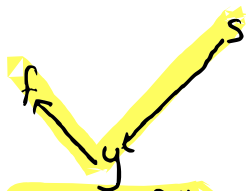
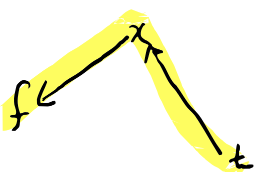


$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial s}$$

Similarly, two paths from t to f,



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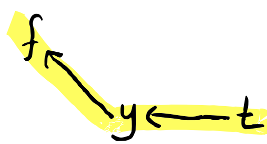

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial t}$$

Example $f(x,y) = x^2 - y^2$

$x(s,t) = s - t \Rightarrow f(x(s,t), y(s,t))$

$y(s,t) = t+1$

$= (s-t)^2 - (t+1)^2$

$= s^2 - 2st + t^2 - t^2 - 2t - 1$

$= s^2 - 2st - 2t - 1$

So $\frac{\partial f}{\partial s} = 2s - 2t$, $\frac{\partial f}{\partial t} = -2s - 2$

On the other hand,

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = -2y$$

$$\frac{\partial x}{\partial s} = 1, \quad \frac{\partial y}{\partial s} = 0, \quad \frac{\partial x}{\partial t} = -1, \quad \frac{\partial y}{\partial t} = 1$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2x = 2(s-t) = 2s-2t.$$

$$\frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} = -2x-2y = -2(s-t) - 2(t+1)$$
$$= -2s-2$$